LESSON 23 - STUDY GUIDE

Abstract. In this lesson we study harmonic and holomorphic functions on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and their relation to data on the boundary ∂D, by the Poisson integral representation. We introduce the Hardy spaces of harmonic and analytic functions on D and examine several of their properties.

1. Harmonic and holomorphic functions on the unit disk, the Poisson integral and Hardy spaces.

Study material: The subject of harmonic and analytic functions on the unit disk is a very important one in itself, in the field of complex analysis. There are many textbooks exclusively dedicated to the subject, that take it far and deep, among which I recommend Kenneth Hoffman's short book on "Banach Spaces of Analytic Functions" [3] or Paul Koosis' "Introduction to H_p spaces". Rudin's "Real and Complex Analysis" [7] also has plenty of material on this beautiful and interesting subject.

But not much is needed for our purposes, and several harmonic analysis books even skip most, if not all of this classical analysis material, to advance by other means straight to the conjugation problem and Marcel Riesz's Theorem on the convergence in L^p norm of Fourier series, for $1 < p < \infty$. Katznelson himself has a full section dedicated to Hardy spaces, **3 - The Hardy Spaces**, from chapter **III - The Conjugate Function and Functions Analytic in the Unit Disk**, but presents it *after* proving the convergence in L^p norm for Fourier series.

So, again, most of the presentation here is my own, gathering what I think are the basic and most fundamental results strictly needed towards the goal of proving that $L^p(\mathbb{T})$ spaces admit conjugation.

In the last lesson we established the important result that the convergence of the symmetric partial sums of Fourier series in $L^p(\mathbb{T})$ norm is equivalent to $L^p(\mathbb{T})$ admitting conjugation.

Recall that the conjugation operator gets its name through a procedure that connects Fourier series to the complex analysis theory of harmonic and holomorphic functions on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. This is the historic origin of the problem and the idea is to start with a function $f \in L^p(\mathbb{T})$ and regard it as the boundary value at ∂D of a harmonic function on D obtained by the Poisson formula, for $|z| = r < 1$,

$$u(z) = u_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|}\hat{f}(n)e^{int} = f * P_r(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\frac{1-r^2}{1-2r\cos(t-s)+r^2}ds,$$

which we know converges to f in the $L^p(\mathbb{T})$ norm as $r \to 1^-$ because the Poisson kernel is an approximate identity. Then, we take the unique harmonic conjugate of u on D that vanishes at the origin, which we also saw last lesson can be written as the convolution with the conjugate Poisson kernel Q_r

$$v(z) = v_r(t) = -i\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) r^{|n|}\hat{f}(n)e^{int} = f * Q_r(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\frac{2\sin(t-s)}{1-2r\cos(t-s)+r^2},$$

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and we finally bring the harmonic conjugate back to the boundary ∂D by making $r = 1$ which makes sense at least as a Fourier multiplier operator acting on f and given by a distribution in $\mathcal{D}'(\mathbb{T})$

$$\tilde{f} = -i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\hat{f}(n)e^{int}.$$

Observe, in fact, that if the conjugate $\tilde{f}$ actually corresponds to a function in $L^1(\mathbb{T})$, with Fourier coefficients thus given by $\hat{\tilde{f}}(n) = -i \operatorname{sgn}(n)\hat{f}(n)$, then the harmonic conjugate function v on D is actually the Poisson integral of $\tilde{f}$,

$$v(z) = v_r(t) = -i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)r^{|n|}\hat{f}(n)e^{int} = \sum_{n=-\infty}^{\infty} r^{|n|}\hat{\tilde{f}}(n)e^{int} = f * Q_r(t) = \tilde{f} * P_r(t).$$

The crucial result, of Theorem 1.7 in the last lesson, is then the fact that Fourier series converge in $L^p(\mathbb{T})$ norm for all functions $f \in L^p(\mathbb{T})$ if and only if $L^p(\mathbb{T})$ admits conjugation, in the sense that whenever f is in $L^p(\mathbb{T})$ than the conjugate $\tilde{f}$ also is. The boundedness of this linear operator, given by the Fourier multiplier $\{-i \operatorname{sgn}(n)\}$, is then automatically guaranteed by the closed graph theorem, as established in Proposition 1.4

Of course this result could have been established directly. It was not, historically, how the conjugate operator arose and leaves the choice of its name unjustified, but one could argue, as we did in the proof of Theorem 1.7, that the uniform boundedness of the operator norm of the symmetric partial sums of the Fourier series in $L^p(\mathbb{T})$ is equivalent to the uniform boundedness of the partial sums of the Riesz projection operator, as they are just frequency shifted from each other, and that can be obtained by multiplication with adequate oscillatory exponentials, so that

$$\|\sum_{-N}^{N} \hat{f}(n)e^{int}\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})} \quad \Leftrightarrow \quad \|\sum_{0}^{2N} \hat{f}(n)e^{int}\|_{L^p(\mathbb{T})} \leq \tilde{C}\|f\|_{L^p(\mathbb{T})},$$

for positive constants $C, \tilde{C}$ independent of N. And this then implies

$$\sum_{-\infty}^{\infty} \hat{f}(n)e^{int} \quad \text{converges in } L^p(\mathbb{T}) \quad \Leftrightarrow \quad \sum_{0}^{\infty} \hat{f}(n)e^{int} \quad \text{converges in } L^p(\mathbb{T}),$$

which in turn is equivalent to $L^p(\mathbb{T})$ admitting Riesz projection, because it is enough to know that for every $f \in L^p(\mathbb{T})$ there exists a function in $Pf \in L^p(\mathbb{T})$ with Fourier series $Pf \sim \sum_0^\infty \hat{f}(n)e^{int}$ to guarantee its convergence, for the partial sum operators will then be just the difference of two shifted projections. And finally noting that, in terms of Fourier coefficients

$$-i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\hat{f}(n)e^{int} \sim -i \left(2\sum_{0}^{\infty} \hat{f}(n)e^{int} - \sum_{-\infty}^{\infty} \hat{f}(n)e^{int} - \hat{f}(0) \right),$$

one concludes that

$$L^p(\mathbb{T}) \text{ admits conjugation} \quad \Leftrightarrow \quad L^p(\mathbb{T}) \text{ admits projection} \quad \Leftrightarrow \quad \sum_{-\infty}^{\infty} \hat{f}(n)e^{int} \text{ converges in } L^p(\mathbb{T})$$

So, by leaving out the middle step of complex analysis one can then try to prove directly that $L^p(\mathbb{T})$ admits conjugation (see, for example Grafakos [2] that presents, in Theorem 4.1.7, such a proof due to Bochner).

The first proof that $L^p(\mathbb{T})$ admits conjugation, for $1 < p < \infty$, and thus that Fourier series converge in these spaces was concluded by Marcel Riesz in 1924 and follows the complex analysis approach. Although we are going to use modern methods, nevertheless we will follow this path as it has great interest and importance in its own right. In particular, it connects harmonic analysis on $\mathbb{T}$ with the theory of spaces of harmonic and analytic functions on the unit disk D, so called Hardy spaces. We will develop a careful characterization of these spaces of functions and their correspondence to the spaces of functions and measures at $\partial D = \mathbb{T}$, in particular pointwise limits as $r \to 1^-$, which will enable us to more accurately understand and study the conjugation operator.

The Poisson kernel is an approximate identity as $r \to 1^-$ and therefore it is important to be able to recognize if a harmonic or holomorphic function on D is obtained by convolution of P_r with an $L^p(\mathbb{T})$ function, because such a kernel will be more helpful to have a better understanding of the functions as one approaches the boundary of the disk D. From the L^p estimates for convolutions and the fact that $\|P_r\|_{L^1(\mathbb{T})} = 1$, for all $r < 1$, we conclude that necessarily

$$\|f * P_r\|_{L^p(\mathbb{T})} \leq \|P_r\|_{L^1(\mathbb{T})} \|f\|_{L^p(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})}, \quad \text{for all} \quad 0 < r < 1,$$

so that all harmonic functions on D that arise as convolutions of $L^p(\mathbb{T})$ functions with the Poisson kernel have uniformly bounded $L^p(\mathbb{T})$ norms on circles at fixed radius. The converse, though, is not necessarily true and to see it one needs to look no further than at the Poisson kernel itself, for $\|P_r\|_{L^1(\mathbb{T})} = 1$ for all $0 < r < 1$ but

$$P_r = \delta * P_r,$$

i.e. P_r is a harmonic function obtained by the convolution of P_r, not with a function, but with a measure. Which is expected, as P_r is an approximate identity when $r \to 1^-$ and converges then to the Dirac-δ at the origin, in the sense of distributions, corresponding to its Fourier coefficients all equal to one when $r = 1$. More generally, if $\mu \in \mathcal{M}(\mathbb{T})$ is any Borel measure on $\mathbb{T}$, one can as well construct a harmonic function on D by convolution with P_r

$$u(z) = u_r(t) = P_r * \mu(t) = \int_{\mathbb{T}} P_r(t-s) d\mu(s),$$

which satisfies

$$\begin{aligned} (1.1) \quad \|u_r\|_{L^1(\mathbb{T})} = \|P_r * \mu\|_{L^1(\mathbb{T})} = \frac{1}{2\pi} \int_{\mathbb{T}} |P_r * \mu(t)|\, dt &= \frac{1}{2\pi} \int_{\mathbb{T}} \left| \int_{\mathbb{T}} P_r(t-s) d\mu(s) \right| dt \\ &\leq \int_{\mathbb{T}} \frac{1}{2\pi} \int_{\mathbb{T}} |P_r(t-s)| dt\, d|\mu|(s) \leq \|P_r\|_{L^1(\mathbb{T})} \int_{\mathbb{T}} d|\mu|(s) = \|\mu\|_{\mathcal{M}(\mathbb{T})}, \end{aligned}$$

for all $0 < r < 1$. Convolutions of the Poisson kernel with $L^1(\mathbb{T})$ functions can therefore be considered as particular cases of these measures, through the identification $f \in L^1(\mathbb{T}) \mapsto \frac{1}{2\pi} f(t)dt \in \mathcal{M}(\mathbb{T})$ with absolutely continuous measures with respect to the Lebesgue measure.

Hardy spaces on D are defined as harmonic, or holomorphic functions, with uniformly bounded $L^p(\mathbb{T})$ norms on circles at fixed radius $r < 1$, such as those seen in these examples.

Definition 1.1. *Let $1 \leq p \leq \infty$. Then, we define the Hardy spaces of harmonic functions on D, and denote them by $h^p(D)$, as*

$$h^p(D) = \Big\{ u : D \to \mathbb{C} \ \ \textit{harmonic} \ : \sup_{0<r<1} \int_{\mathbb{T}} |u_r(t)|^p dt < \infty \Big\},$$

with norm

$$\|u\|_{h^p(D)} = \sup_{0<r<1} \|u_r\|_{L^p(\mathbb{T})}.$$

Similarly, we define the Hardy spaces of holomorphic functions $H^p(D)$, as

$$H^p(D) = \Big\{ f : D \to \mathbb{C} \ \ \textit{holomorphic} \ : \sup_{0<r<1} \int_{\mathbb{T}} |f_r(t)|^p dt < \infty \Big\},$$

with norm

$$\|f\|_{H^p(D)} = \sup_{0<r<1} \|f_r\|_{L^p(\mathbb{T})}.$$

Because every holomorphic function is harmonic, we obviously have $H^p(D) \subset h^p(D)$ for all p. And because $\mathbb{T}$ has finite measure, due to the usual inclusion of $L^p(\mathbb{T})$ spaces we also have $h^q(D) \subset h^p(D)$ or $H^q(D) \subset H^p(D)$ for $q \geq p$, with the corresponding inequality for the norms.

And from the discussion that precedes this definition, if $\mu \in \mathcal{M}(\mathbb{T})$ then

$$P_r * \mu \in h^1(D),$$

in particular $P_r = P_r * \delta \in h^1(D)$, while, if $f \in L^p(\mathbb{T})$, then

$$P_r * f \in h^p(D).$$

Recall from the last lesson, that every holomorphic function f on D, being represented by a Taylor series of radius of convergence at least one, can always be written, for $r < 1$, as

$$f(z) = f_r(t) = \sum_{n=0}^{\infty} a_n r^n e^{int}, \tag{1.2}$$

for some complex sequence $\{a_n\}_{n=0,1,2,\ldots}$ (that corresponds to $a_n = f^{(n)}(0)/n!$), with convergence holding absolutely and uniformly for every fixed radius, and thus $\widehat{f_r}(n) = a_n r^n$ for $n \geq 0$, while $\widehat{f_r} = 0$ for $n < 0$. Analogously, every harmonic function u on D can always be written, for $r < 1$, as

$$u(z) = u_r(t) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{int}, \tag{1.3}$$

for some unique complex sequence $\{c_n\}_{n\in\mathbb{Z}}$ with $\widehat{u_r}(n) = c_n r^{|n|}$ for $n \in \mathbb{Z}$. The problem is that, whereas in the last lesson we started a priori by assuming that we had holomorphic and harmonic functions up to, *and including*, the boundary ∂D, so that we had from the start well defined and continuous functions at $r = 1$, if we start instead by assuming only these properties on the interior of the disk D only, then things can get in general quite wild as the radius approaches $r \to 1^-$. From a Fourier series point of view, the problem clearly has to do with the nature of the coefficients $\{a_n\}_{n=0,1,2,\ldots}$ and $\{c_n\}_{n\in\mathbb{Z}}$, which we would like to be the Fourier coefficients of the boundary functions, when $r = 1$, but that in full generality cannot even be easily ensured to grow polynomially in n in order to correspond to arbitrary distributions.

However, if a harmonic function on D can indeed be represented as the convolution of the Poisson kernel with a distribution, then that representation is unique. This, of course, includes $L^p(\mathbb{T})$ functions and measures $\mathcal{M}(\mathbb{T})$ as particular cases.

Proposition 1.2. *Let $u : D \to \mathbb{C}$ be a harmonic or, particularly, a holomorphic function. Then, if there exists a distribution $F \in \mathcal{D}'(\mathbb{T})$ such that $u = P_r * F$ then F is unique.*

Proof. From the general representation formula for harmonic functions on D (1.3) and its absolute convergence for $r < 1$, we know that its Fourier coefficients are $\widehat{u_r}(n) = c_n r^{|n|}$ for all $n \in \mathbb{Z}$. On the other hand, from Property (3) in Proposition 1.1, of Lesson 21, concerning the Fourier coefficients of the convolutions of distributions and $L^1(\mathbb{T})$ functions, we know that $\widehat{P_r * F}(n) = \widehat{P_r}(n)\widehat{F}(n) = r^{|n|}\widehat{F}(n)$, so that, if $u = P_r * F$ on D then necessarily $c_n r^{|n|} = r^{|n|}\widehat{F}(n)$ and thus $c_n = \widehat{F}(n)$. And by the uniqueness of Fourier coefficients for distributions, Corollary 1.4, in Lesson 21, we conclude that F is unique. $\square$

But even though the coefficients $\{a_n\}_{n=0,1,2,\ldots}$ and $\{c_n\}_{n\in\mathbb{Z}}$ might be troublesome by themselves, once multiplied by powers of the radius $r < 1$ they immediately become absolutely summable and, in particular, fixing any radius, say $R < 1$, one can always write these functions for smaller radii $r < R$ as a Poisson integral considering the boundary at R, for the ratio of the radii $\rho = r/R$,

$$f_r(t) = \sum_{n=0}^{\infty} a_n r^n e^{int} = \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^n R^n e^{int} = P_{\frac{r}{R}} * f_R(t) = P_\rho * f_R(t),$$

and analogously

$$u_r(t) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{int} = \sum_{n=-\infty}^{\infty} c_n \left(\frac{r}{R}\right)^{|n|} R^n e^{int} = P_{\frac{r}{R}} * u_R(t) = P_\rho * u_R(t).$$

This leads to a simple, but important, conclusion regarding Hardy functions. For any $1 \le p \le \infty$, we have,

$$\|f_r\|_{L^p(\mathbb{T})} = \|P_\rho * f_R\|_{L^p(\mathbb{T})} \le \|P_\rho\|_{L^1(\mathbb{T})} \|f_R\|_{L^p(\mathbb{T})} = \|f_R\|_{L^p(\mathbb{T})},$$

and

$$\|u_r\|_{L^p(\mathbb{T})} = \|P_\rho * u_R\|_{L^p(\mathbb{T})} \le \|P_\rho\|_{L^1(\mathbb{T})} \|u_R\|_{L^p(\mathbb{T})} = \|u_R\|_{L^p(\mathbb{T})},$$

for $r < R$, because $\|P_\rho\|_{L^1(\mathbb{T})} = 1$. We have thus established the following.

Proposition 1.3. *Let $u \in h^p(D)$ for some $1 \le p \le \infty$. Then, its $L^p(\mathbb{T})$ norms over circles at fixed radius $\|u_r\|_{L^p(\mathbb{T})}$ are a nondecreasing function of r. And thus*

$$\|u\|_{h^p(D)} = \sup_{0<r<1} \|u_r\|_{L^p(\mathbb{T})} = \lim_{r \to 1^-} \|u_r\|_{L^p(\mathbb{T})}.$$

The same holds for analytic functions in $H^p(D)$, a subspace of $h^p(D)$.

Although it is simply a matter of direct application of convolution inequalities, as we did above, to show that harmonic functions defined by Poisson integrals have $L^p(\mathbb{T})$ norms uniformly bounded for any $0 < r < 1$, and thus are in $h^p(D)$, the reverse conclusion is more complicated. The problem is that boundedness of sequences, or families of functions, in $L^p(\mathbb{T})$, does not guarantee the existence of convergent subsequences, which is what is needed to obtain a limit function at the boundary, as $r \to 1^-$. As we saw, back in Lesson 9, closed balls are not compact in infinite dimensional normed spaces, to enable us to extract convergent subsequences. But they are compact in the weak* topology, due to the Banach-Alaoglu theorem. So as long as a space can be recognized as a dual space of another one, we can get weak* convergence of some subsequence, which will then be enough to use with the exponentials playing the role of the test functions, to obtain the limiting Fourier coefficients at $r = 1$. Recall that the $L^p(\mathbb{T})$ spaces are dual of $L^{p'}(\mathbb{T})$ only for $1 < p \le \infty$, but that $L^1(\mathbb{T})$ is no one's dual, so that it was left out of the application of the Banach-Alaoglu theorem, in Theorem 2.6 of Lesson 9. Nevertheless, the space of complex Borel measures $\mathcal{M}(\mathbb{T})$ is the dual of the space of continuous functions $C(\mathbb{T})$ (see, for example, Corollary 7.18 in Folland's book [1]) and $C(\mathbb{T})$ is separable (the polynomials with rational coefficients form a dense countable subset), so that Proposition 2.5 in Lesson 9 also guarantees that bounded sequences of measures have weak* converging subsequences, just like in Theorem 2.6 for $L^p(\mathbb{T})$ spaces. So that, concerning $L^1(\mathbb{T})$ functions, the only thing that can be done is to identify them through their embedding $f \in L^1(\mathbb{T}) \mapsto \frac{1}{2\pi} f(t)dt \in \mathcal{M}(\mathbb{T})$ with the subspace of absolutely continuous measures with respect to the Lebesgue measure in $\mathbb{T}$, and to use the Banach-Alaoglu theorem for the whole space of measures.

We therefore have the fundamental theorem characterizing the Hardy spaces of harmonic functions.

Theorem 1.4. *Let $u : D \to \mathbb{C}$ be harmonic. Then, we have*

1. *$u \in h^1(D) \Leftrightarrow u = P_r * \mu$ for $\mu \in \mathcal{M}(\mathbb{T})$.*
2. *For $1 < p \le \infty$, $u \in h^p(D) \Leftrightarrow u = P_r * f$ for $f \in L^p(\mathbb{T})$.*
3. *u_r converges in the $L^1(\mathbb{T})$ norm as $r \to 1^- \Leftrightarrow u = P_r * f$ for $f \in L^1(\mathbb{T})$.*

(4) *u_r converges uniformly as $r \to 1^-$ $\Leftrightarrow u = P_r * f$ for $f \in C(\mathbb{T})$.*

Proof.

(1) That $P_r * \mu \in h^1(D)$ when $\mu \in \mathcal{M}(\mathbb{T})$ was seen above, in (1.1). Conversely, let $u \in h^1(D)$. Then, identifying u_r with the measure $\mu_r = \frac{1}{2\pi} u_r(t)dt$ we have $\|u_r\|_{L^1(\mathbb{T})} = \|\mu_r\|_{\mathcal{M}(\mathbb{T})} \leq \|u\|_{h^1(D)}$. Now, using the fact that the space of continuous functions $C(\mathbb{T})$ is separable while its dual is the space of Borel measures $C(\mathbb{T})' = \mathcal{M}(\mathbb{T})$, we can use Proposition 2.5 in Lesson 9 to extract a sequence μ_{r_j}, with $r_j \to 1^-$ as $j \to \infty$, which is weak* convergent in $\mathcal{M}(\mathbb{T})$. Therefore, this means that there exists $\mu \in \mathcal{M}(\mathbb{T})$ such that, for every $\phi \in C(\mathbb{T})$ we have

$$\langle \mu_{r_j}, \phi \rangle = \int_{\mathbb{T}} \phi \, d\mu_{r_j} \to \int_{\mathbb{T}} \phi \, d\mu = \langle \mu, \phi \rangle,$$

as $j \to \infty$. And making $\phi = e^{-int}$ we thus conclude

$$\widehat{\mu_{r_j}}(n) \to \widehat{\mu}(n),$$

for every $n \in \mathbb{Z}$, as $j \to \infty$. But from the general representation formula for harmonic functions on D (1.3) we know that $\widehat{\mu_{r_j}}(n) = c_n r_j^{|n|}$ from which we conclude, because $r_j \to 1$, that $c_n = \widehat{\mu}(n)$ and therefore, for $r < 1$,

$$u(z) = u_r(t) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{int} = \sum_{n=-\infty}^{\infty} \widehat{\mu}(n) r^{|n|} e^{int} = P_r * \mu(t).$$

(2) The proof is exactly the same as in (1), except that the Banach-Alaoglu theorem is applied to $L^p(\mathbb{T})$ as the dual of $L^{p'}(\mathbb{T})$, more specifically as was done in Theorem 2.6 of Lesson 9.

(3) and (4) follow easily, by just using the fact that P_r is an approximate identity.

□

Several interesting observations now follow. The first one is that the unique solution $u \in C^2(D) \cap C(\overline{D})$ to the Dirichlet problem for the Laplace equation on D

$$\Delta u = 0, \quad u_{|\partial D} = f,$$

with $f \in C(\mathbb{T})$, is given by the Poisson integral $u = P_r * f$ which is harmonic in D, and converges uniformly to f as $r \to 1^-$. One might wonder if there could exist a different harmonic solution to this Dirichlet problem, not given by a Poisson integral, which, from Property (4) above, would only be possible if the solution did not converge uniformly to f as $r \to 1^-$. But the maximum principle for harmonic functions guarantees uniqueness, so the only solution of the Dirichlet problem for the disk D is always given by a Poisson integral and thus always converges uniformly to the continuous boundary data as $r \to 1^-$.

A second observation is that it is an easy exercise to show that, for $u \in h^1(D)$ we have

$$\|u\|_{h^1(D)} = \lim_{r \to 1^-} \|u_r\|_{L^1(\mathbb{T})} = \lim_{r \to 1^-} \|P_r * \mu\|_{L^1(\mathbb{T})} = \|\mu\|_{\mathcal{M}(\mathbb{T})},$$

and for $1 < p \leq \infty$,

$$\|u\|_{h^p(D)} = \lim_{r \to 1^-} \|u_r\|_{L^p(\mathbb{T})} = \lim_{r \to 1^-} \|P_r * f\|_{L^p(\mathbb{T})} = \|f\|_{L^p(\mathbb{T})},$$

so that Theorem 1.4, parts (1) and (2), basically show that the Poisson integrals provide linear isometric bijections between $h^1(D)$ harmonic functions in the interior of the disk D and measures at the boundary $\mathcal{M}(\mathbb{T})$, on the one hand, and $h^p(D)$ and $L^p(\mathbb{T})$, on the other, for $1 < p \leq \infty$.

A third, and final important observation, is that case (2) for $p = \infty$ actually characterizes the bounded harmonic functions on D: they are the Poisson integrals of (boundary) functions in $L^\infty(\mathbb{T})$.

The conjugation problem, for $1 < p < \infty$, which we seek to establish the convergence in $L^p(\mathbb{T})$ norm of Fourier series, can also now be stated in terms of Hardy spaces. The question then is, given any $u \in h^p(D)$, will its harmonic conjugate v on D, with $v(0) = 0$, satisfy $v \in h^p(D)$?

As the harmonic Hardy spaces end up just being harmonic representations in D of $\mathcal{M}(\mathbb{T})$ or $L^p(\mathbb{T})$ at the boundary, it is more interesting to look at particular subspaces. An interesting case is that of real positive harmonic functions.

Theorem 1.5. (Herglotz) *Let $u : D \to \mathbb{R}$ be harmonic. Then u is nonnegative if and only if $u = P_r * \mu$ with $\mu \in \mathcal{M}(\mathbb{T})$ a positive measure.*

Proof. If μ is a positive measure and $u = P_r * \mu$ then u is obviously harmonic and nonnegative on D. Conversely, if u is harmonic and nonnegative, then for every $r < 1$ we have

$$\|u_r\|_{L^1(\mathbb{T})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u_r(t)|dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_r(t)dt.$$

But from the mean value theorem for harmonic functions, this last integral is the value of u at the center of the disk, i.e. $u(0)$. So we have $\|u_r\|_{L^1(\mathbb{T})} = u(0)$ for all $0 < r < 1$ which implies $u \in h^1(D)$. And using Part (1) of Theorem 1.4 we conclude that $u = P_r * \mu$ for some measure that necessarily must be positive. □

Of course, the most interesting subspaces of $h^p(D)$ are the Hardy spaces of holomorphic functions $H^p(D)$. By combining Theorem 1.4 with the general representation of holomorphic functions on D, based on the expansion in Taylor series, we can conclude that for $1 < p \leq \infty$ we have

$$f \in H^p(D) \Leftrightarrow u = P_r * F \quad \text{for} \quad F \in L^p(\mathbb{T})$$

but with the extra property that the Fourier coefficients satisfy $\hat{F}(n) = 0$, for $n < 0$. Also

$$f \in H^1(D) \Leftrightarrow u = P_r * \mu \quad \text{for} \quad \mu \in \mathcal{M}(\mathbb{T}),$$

with $\hat{\mu}(n) = 0$ for $n < 0$. But it is here that a very important difference occurs between harmonic and holomorphic functions, because a fundamental theorem by Marcel and Frigyes Riesz (the only joint result by the two famous brothers) states that measures that represent holomorphic functions in $H^1(D)$ by their Poisson integral are always absolutely continuous with respect to the Lebesgue measure.

Theorem 1.6. (F. and M. Riesz) *Let $\mu \in \mathcal{M}(\mathbb{T})$ such that $\hat{\mu}(n) = 0$ for $n < 0$. Then, $\mu \ll dt$, i.e. there exists $F \in L^1(\mathbb{T})$, with $\hat{F}(n) = 0$ for $n < 0$, such that $\mu = \frac{1}{2\pi}F(t)dt$.*

We will not prove this theorem here as it would take us a bit far off our path. But it can be found in Katnzelson [4, 5] in section **3 - The Hardy Spaces**, from chapter **III - The Conjugate Function and Functions Analytic in the Unit Disk**.

So we conclude that, for the holomorphic Hardy spaces $H^p(D)$, for any $1 \leq p \leq \infty$, functions are always represented by Poisson integrals with $L^p(\mathbb{T})$ functions, whose Fourier coefficients vanish for negative frequencies.

In the next lesson we will start developing methods to study pointwise limits of these functions on D as we approach the boundary $r \to 1^-$.

References

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